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Averages and K-Functionals Related to the Laplacian

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For R^d or T^d, averages on balls and spheres are shown to satisfy an equivalence relation with K-functionals that are generated by the Laplacian. The converse result is given in terms of strong converse inequality of type A and type D for the averages on the Ball and on the sphere, respectively. Combinations of averages on concentric balls and spheres yield strong converse results of type B for higher levels of smoothness.

1. INTRODUCTION

In this section we introduce concepts used in the paper and describe some of the main results. We will also mention some earlier related results.

The smoothness of elements of Banach space B of functions (or distributions) on R^d or on T^d is described by the K-functional $K_{\Delta,\ell}(f, t^{2\ell})_B$ is given by

$$K_{\Delta,\ell}(f, t^{2\ell})_B = \inf_{g \in J(\ell)} (\|f-g\|_B + t^{2\ell} \|\Delta^\ell g\|_B,) \tag{1.1}$$

where Δf is the Laplacian, given by $\Delta f = \partial^2 f/\partial x_1^2 + \cdots + \partial^2 f/\partial x_d^2$, $\Delta^\ell f = \Delta(\Delta^{\ell-1} f)$, and $J(\ell)$ is an appropriate class of functions described in the theorems in which the K-functional is used. As it will turn out, a wide choice of classes $J(\ell)$ will lead to the same K-functional.

The averaging operator (on R^d or on T^d), $B_t(f, x)$, is given for a locally integrable function by

$$B_t(f, x) = \frac{1}{m(B)\, t^d} \int_{Bt} f(x+u)\, dV(u), \tag{1.2}$$

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where $m(B)$ is the volume of the unit ball in R^d (or T^d), $Bt = \{x: |x| \leqslant t\}$ (which implies $m(Bt) = m(B)\, t^d$), and $dV(u)$ is the Lebesgue measure element in R^d (or T^d). The averaging operator, $S_t(f, x)$, is given for locally integrable functions (a.e. in x) by

$$S_t(f, x) = \frac{1}{m(S)\, t^{d-1}} \int_{St} f(x+u)\, d\sigma(u), \tag{1.3}$$

where $m(S)$ is the measure of the unit sphere in R^d (the "surface" area), $St = \{x: |x| = t\}$ (which implies $m(St) = m(S)\, t^{d-1}$), and $d\sigma(u)$ is the measure element ($d-1$ dimensional) on S. Sometimes when x is understood from the context we write $B_t f$ and $S_t f$ for $B_t(f, x)$ and $S_t(f, x)$.

The basic relation between the rate of approximation of f by $B_t f$ and by $S_t f$ and the K-functional is given by

$$\|B_t f - f\|_B \approx K_{\Delta, 1}(f, t^2)_B \equiv K_\Delta(f, t^2)_B \tag{1.4}$$

and

$$\sup_{0<h\leqslant t} \|S_h f - f\|_B \approx K_\Delta(f, t^2)_B, \tag{1.5}$$

where $\varphi(t) \approx \psi(t)$ means that there exists a constant C for which $C^{-1}\varphi(t) \leqslant \psi(t) \leqslant C\varphi(t)$. The direct and converse results of (1.4) and (1.5) are the estimate of the left hand side (of either (1.4) or (1.5)) by the right hand side and vice versa, respectively. The converse results in (1.4) and (1.5) are of type A and type D, respectively in the classification introduced and discussed in [Di-Iv].

For $B = L_p(R^d)$ or $L_p(T^d)$ with $1 < p < \infty$, (1.5) is essentially known. The equivalence (1.4) is new. Earlier, equivalence like (1.5) was achieved by the first author [Di, II] for

$$A_h f = \sum_{i=1}^{d} (f(x + e_i h) + f(x - e_i h))$$

for any orthonormal set e_i in R^d (which would imply the converse relation in (1.5)). Equivalence like (1.4) was achieved for an average on the box with center at x in [Di-Iv]. The present technique (which is different in many respects from the above mentioned results) leads to equivalences between approximation of f by combinations of $B_{kt} f$ or $S_{kt} f$ and the K-functional $K_{\Delta, \ell}(f, t^{2\ell})$. The converse part of those will be of type B or D respectively (using combinations of $B_{kt} f$ and $S_{kt} f$). These results have advantage over iterations, as the expressions estimated are simpler. In case of estimate by S_t (see (1.5)), iteration would lead to many suprema on different h_j.

Further discussions and comparisons will be carried out when our results are established.

2. AVERAGES AND THE LAPLACIAN

In this section, we achieve some relations between the rate of approximation of $f(x)$ by $B_t(f, x)$ and by $S_t(f, x)$ and the Laplacian of f. These results will be crucial for the direct and the converse estimates.

THEOREM 2.1. *Suppose $f \in C^2$ locally (f does not need to be bounded on R^d). Then, for $S_t(f, x)$ and $B_t(g, x)$, given by* (1.3) *and* (1.2), *we have*

$$S_t(f, x) - f(x) = \frac{m(B)}{m(S)} \int_0^t \tau B_\tau(\Delta f, x)\, d\tau$$

$$= \frac{1}{d} \int_0^t \tau B_\tau(\Delta f, x)\, d\tau. \tag{2.1}$$

Proof. We may write for $d \geqslant 2$

$$S_t(f, x) - f(x) = \frac{1}{m(S)\, t^{d-1}} \int_{St} (f(x+u) - f(x))\, d\sigma(u)$$

$$= \frac{1}{m(S)} \int_S (f(x+tu) - f(x))\, d\sigma(u)$$

$$= \frac{1}{m(S)} \int_S \int_0^t \frac{\partial}{\partial \tau} f(x+\tau u)\, d\tau\, d\sigma(u)$$

$$= \frac{1}{m(S)} \int_0^t \left\{ \int_S \frac{\partial}{\partial \tau} f(x+\tau u)\, d\sigma(u) \right\} d\tau$$

$$= \frac{1}{m(S)} \int_0^t \tau^{-d+1} \int_{S\tau} \frac{\partial}{\partial n} f(x+w)\, d\sigma_\tau(w)\, d\tau$$

(using the divergence theorem)

$$= \frac{1}{m(S)} \int_0^t \tau^{-d+1} \int_{B\tau} \Delta f(x+w)\, dV(w)$$

$$= \frac{m(B)}{m(S)} \int_0^t \tau B_\tau(\Delta f, x)\, d\tau$$

$$= \frac{1}{d} \int_0^t \tau B_\tau(\Delta f, x)\, d\tau.$$

For $d=1$, $S_t(f,x)=\frac{1}{2}(f(x+t)+f(x-t))$, $B_t(f,x)=(1/2t)\int_{-t}^{t} f(x+\tau)\,d\tau$, and $\Delta f=f''(x)$, in which case direct calculation yields (2.1) as well. ∎

THEOREM 2.2. *Suppose $f\in C^2$ locally. Then,*

$$B_t(f,x)-f(x)=\frac{1}{t^d}\int_0^t \tau^{d-1}\int_0^\tau \eta B_\eta(\Delta f,x)\,d\eta\,d\tau. \tag{2.2}$$

Proof. We write

$$\begin{aligned}B_t(f,x)-f(x)&=\frac{1}{m(B)\,t^d}\int_{Bt}(f(x+u)-f(x))\,dV(u)\\&=\frac{1}{m(B)\,t^d}\int_0^t\int_{S\tau}(f(x+u)-f(x))\,d\sigma(u)\,d\tau\\&=\frac{1}{t^d}\int_0^t \tau^{d-1}\int_0^\tau \eta B_\eta(\Delta f,x)\,d\eta\,d\tau.\end{aligned}$$ ∎

We denote, by O_t the operator (on functions of τ with parameter x)

$$O_t(g_\tau(x))=\frac{1}{t^d}\int_0^t \rho^{d-1}\int_0^\rho \tau g_\tau(x)\,d\tau\,d\rho. \tag{2.3}$$

Using this notation, we can use Theorems 2.1 and 2.2 to obtain the following result.

THEOREM 2.3. *For $f\in C^{2\ell+2}$, we have*

$$\begin{aligned}B_t(f,x)-f(x)-\sum_{j=1}^{\ell}\frac{t^{2j}\Delta^j f(x)}{2^j j!\,(d+2)\cdots(d+2j)}\\=O_t(O_{t_1}(O_{t_2}\cdots(O_{t_\ell}(B_\tau(\Delta^{2\ell+2}f,x)))\cdots)),\end{aligned} \tag{2.4}$$

and

$$\begin{aligned}S_t(f,x)-f(x)-\sum_{j=1}^{\ell}\frac{t^{2j}\Delta^j f(x)}{2^j j!\,d\cdots(d+2(j-1))}\\=\frac{1}{d}\int_0^t t_1\,O_{t_1}(O_{t_2}\cdots(O_{t_\ell}(B_\tau(\Delta^{\ell+1}f,x))\cdots))\,dt_1.\end{aligned} \tag{2.5}$$

Proof. The proof is computational and follows by the use of induction as well as estimate of

$$B_\tau(\Delta^j f,x)-\Delta^j f(x)$$

using (2.2) and the identity

$$O_t(\tau^{2j}) = \frac{1}{2j+2} \frac{1}{d+2j+2} t^{2j+2} \qquad \text{for} \quad j=0, 1, ...$$

To obtain (2.5), we use also the estimate (2.1). ∎

In fact, (2.3) is helpful so that we do not have to write so many integration signs. Without (2.3), (2.4) will take the form

$$B_t f(x) - f(x) - \sum_{j=1}^{\ell} \frac{t^{2j} \Delta^j f(x)}{2^j j!\,(d+2) \cdots (d+2j)}$$
$$= \frac{1}{t^d} \int_0^t \rho^{d-1} \int_0^{\rho} t_1^{-d+1} \int_0^{t_1} \rho_1^{d-1} \cdots \int_0^{t_\ell} \rho_\ell^{d-1}$$
$$\times \int_0^{\rho_\ell} \tau B_\tau(\Delta^{\ell+1} f, x)\, d\tau\, d\rho\, d\rho_1 \cdots d\rho_\ell\, dt_1 \cdots dt_\ell, \tag{2.4}'$$

and a similar expression can replace (2.5).

To treat higher degrees of smoothness, we define the operators $B_{\ell,t}(f, x)$ and $S_{\ell,t}(f, x)$ which are essentially combinations of $B_{jt}(f, x)$ and $S_{jt}(f, x)$, respectively. The operators $B_{\ell,t}(f, x)$ and $S_{\ell,t}(f, x)$ are given by

$$B_{\ell,t}(f, x) = \frac{-2}{\binom{2\ell}{\ell}} \sum_{j=1}^{\ell} (-1)^j \binom{2\ell}{\ell-j} B_{jt}(f, x) \tag{2.6}$$

and

$$S_{\ell,t}(f, x) = \frac{-2}{\binom{2\ell}{\ell}} \sum_{j=1}^{\ell} (-1)^j \binom{2\ell}{\ell-j} S_{jt}(f, x), \tag{2.7}$$

respectively.

For the direct estimate, we will need the following result.

THEOREM 2.4. *If f has 2ℓ continuous derivatives, then*

$$B_{\ell,t}(f, x) - f(x) = \frac{2}{\binom{2\ell}{\ell}} \sum_{j=1}^{\ell} (-1)^j \binom{2\ell}{\ell-j}$$
$$\times O_{jt}(O_{t_1}(O_{t_2} \cdots (O_{t_{\ell-1}}(B_\tau(\Delta^\ell f, x)) \cdots))) \tag{2.8}$$

and

$$S_{\ell,t}(f,x)-f(x)=\frac{2}{d\binom{2\ell}{\ell}}\sum_{j=1}^{\ell}(-1)^j\binom{2\ell}{\ell-j}$$

$$\times\int_0^{jt}t_1\,O_{t_1}(O_{t_2}\cdots(O_{t_{\ell-1}}(B_\tau(\Delta^\ell f,x))\cdots))\,d\tau \quad (2.9)$$

Proof. We recall that for $g(t)\in C^{2\ell}$,

$$\begin{aligned}\vec{\Delta}_1^{2\ell}g(t)&\equiv\sum_{m=0}^{2\ell}\binom{2\ell}{m}(-1)^m\,g(t+m)\\&=(-1)^\ell\sum_{j=-\ell}^{\ell}\binom{2\ell}{\ell-j}(-1)^j\,g(t+\ell-j)\\&=g^{(2\ell)}(\theta_\ell)\end{aligned}$$

with $0<\theta_\ell<2\ell$. Setting $g(t)=(t-\ell)^{2r}$ with r integer satisfying $0<r<\ell$, and using (2.4) and (2.5) with ℓ_1 (there) satisfying $\ell_1=\ell-1$, we conclude the proof of our theorem. ▮

THEOREM 2.5. *For a function f with $2\ell+2$ derivatives locally, we have*

$$B_{\ell,t}(f,x)-f(x)-\frac{t^{2\ell}}{2^2}\frac{(-1)^\ell\,\ell!\,\Delta^\ell f(x)}{(d+2)\cdots(d+2\ell)}$$

$$=\frac{2}{\binom{2\ell}{\ell}}\sum_{j=1}^{\ell}(-1)^{j-1}\binom{2\ell}{\ell-j}O_{jt}(O_{t_1}(O_{t_2}\cdots(O_{t_\ell}(B_\tau(\Delta^{\ell+1}f,x))\cdots))). \quad (2.10)$$

Remark. One may obtain an analogue of (2.10) for $S_{\ell,t}f$, however, the estimate (2.10) is used to prove a strong converse inequality of type A or B (see [Di-Iv]) for $B_{\ell,t}f-f$, and this will not be possible for $S_{\ell,t}f-f$. (For $S_{\ell,t}f-f$, such a relation fails already for $d=1$.) For $S_{\ell,t}f-f$, a strong converse inequality of type D will be achieved as a result of the converse estimate for $B_{\ell,t}f-f$.

Proof. We use (2.4) of Theorem 2.3 with the same ℓ as in our theorem and set $g(t)=(t-\ell)^{2\ell}$ for $g(t)$ given in the proof of Theorem 2.4, and as $g^{(2\ell)}(\theta)=(2\ell)!$, we obtain (2.10). ▮

3. THE DIRECT RESULT $C(R^D)$ AND $C(T^D)$

We will prove first several estimates crucial for the proof of the direct and the converse results. These estimates will be proved first for functions with sufficiently many continuous derivatives. From these, we will deduce them for various Banach spaces of functions or distributions. We now state the estimates for general space B to avoid serious duplication. As the operators $B_t f$ and $S_t f$ were defined only for locally integrable functions, the inequality stated below makes sense, for the present, only for such function spaces. Later, we will extend the definitions of $B_t f$ and $S_t f$ and prove the validity of the inequalities for some other cases.

The estimates with which we will deal are:

$$\|B_t f - f\|_B \leqslant \frac{t^2}{2(d+2)} \|\Delta f\|_B, \tag{3.1}$$

$$\|S_t f - f\|_B \leqslant \frac{t^2}{2d} \|\Delta f\|_B, \tag{3.2}$$

$$\left\| B_t f - f - \sum_{j=1}^{\ell} \frac{t^{2j} \Delta^j f}{2^j j!\,(d+2)\cdots(d+2j)} \right\|_B \leqslant \frac{t^{2\ell+2} \|\Delta^{\ell+1} f\|_B}{2^{\ell+1}(\ell+1)!\,(d+2)\cdots(d+2\ell+2)}, \tag{3.3}$$

$$\left\| S_t f - f - \sum_{j=1}^{\ell} \frac{t^{2j} \Delta^j f}{2^j j!\, d\cdots(d+2(j-1))} \right\|_B \leqslant \frac{t^{2\ell+2} \|\Delta^{\ell+1} f\|_B}{2^{\ell+1}(\ell+1)!\, d\cdots(d+2\ell)}, \tag{3.4}$$

$$\|B_{\ell,t} f - f\|_B \leqslant C(\ell, d)\, t^{2\ell} \|\Delta^\ell f\|_B, \tag{3.5}$$

$$\|S_{\ell,t} f - f\|_B \leqslant C(\ell, d) \frac{d+2\ell}{\ell} t^{2\ell} \|\Delta^\ell f\|_B \tag{3.6}$$

and

$$\left\| B_{\ell,t} f - f - \frac{(-1)^\ell t^{2\ell} \ell!\, \Delta^\ell f}{(2^\ell (d+2)\cdots(d+2\ell)!)} \right\|_B \leqslant C_1(\ell, d)\, t^{2\ell+2} \|\Delta^{\ell+1} f\|_B, \tag{3.7}$$

where

$$C(\ell, d) = \frac{1}{\binom{2\ell}{\ell}} \cdot \frac{1}{2^{\ell-1}\ell!\,(d+2)\cdots(d+2\ell)} \sum_{j=1}^{\ell} j^{2\ell} \binom{2\ell}{\ell-j} \tag{3.8}$$

and

$$C_1(\ell, d) = \frac{1}{\binom{2\ell}{\ell}} \cdot \frac{1}{2^{\ell}(\ell+1)!\,(d+2)\cdots(d+2\ell+2)} \sum_{j=1}^{\ell} j^{2\ell+2} \binom{2\ell}{\ell-j}.$$

We note that (3.1) and (3.2) could be construed as special cases of (3.3) and (3.4) (for $\ell = 0$), and they are special cases of (3.5) and (3.6) (for $\ell = 1$), but, being one of the basic building blocks of this paper, we stated them separately.

THEOREM 3.1. *For $f \in C^2(R^d)$ or $f \in C^2(T^d)$, (3.1) and (3.2) are valid with $B = C(R^d)$ or $C(T^d)$, respectively. For $f \in C^{2\ell+2}(R^d)$ of $f \in C^{2\ell+2}(R^d)$, (3.3), (3.4), and (3.7) are valid with $B = C(R^d)$ or $B = C(T^d)$. For $f \in C^{2\ell}(R^d)$ or $f \in C^{2\ell}(T^d)$, (3.5) and (3.6) are valid with $B = C(R^d)$ or $B = C(T^d)$. Moreover, each time $f \in C^{2m}(R^d)$ is required, it may be replaced by $f \in C^{2m}_{loc}(R^d)$ and $\Delta^m f \in C(R^d)$.*

Proof. Using Theorems 2.1 and 2.2, we obtain (3.1) and (3.2) when we recall that $\|B_\tau g\|_C \leqslant \|g\|_C$ and that $(1/t^d) \int_0^t \tau^{d-1} \int_0^\tau \eta \, d\eta = t^2/2(d+2)$ and $(1/d) \int_0^t \tau \, d\tau = t^2/2d$. Using Theorem 2.3, formula (2.4) and (2.5) and recalling again that $\|B_\tau g\|_C \leqslant \|g\|_C$, the identities

$$O_t(O_{t_1}(O_{t_2} \cdots (O_{t_\ell}(1)) \cdots)) = \frac{t^{2\ell+2}}{2^{\ell+1}(\ell+1)!\,(d+2)\cdots(d+2\ell+2)}$$

and

$$\frac{1}{d}\int_0^t t_1 O_t(O_{t_2} \cdots (O_{t_\ell}(1)) \cdots)\, dt_1 = \frac{t^{2\ell+2}}{2^{\ell+1}(\ell+1)!\, d(d+2)\cdots(d+2\ell)}$$

(as $O_t(\tau^{2j}) = (1/2j+2)(1/d+2j+2)\, t^{2j+2}$) yield our result. To prove (3.5), (3.6), and (3.7) (the latter under the condition $f \in C^{2\ell+2}$ rather than $f \in C^{2\ell}$), we use (2.8), (2.9), and (2.10) and the consideration above to reduce the proof to verification of

$$\|O_{jt} O_{t_1} O_{t_2} \cdots O_{t_{\ell-1}}(B_\tau(\Delta^\ell f))\| \leqslant \frac{t^{2\ell} j^{2\ell}}{2^\ell \ell!\,(d+2)\cdots(d+2\ell)} \|\Delta^\ell f\|$$

and similar estimates. We note that we use only the exact estimate of (3.7) for $\ell = 1$ and low dimension d and hence we did not feel justified in giving detailed computation here.

As the result uses local estimates only, the last part of the theorem is self-evident. In fact, even if f is not bounded, the estimate (3.1), for example, is valid. In two dimensions, we take for example $f(x, y) = x^2 + y^2$, $\Delta f(x, y) = 4$ and $\|B_t f - f\|_{C(R^d)} \leqslant t^2/2$ to illustrate this point. ∎

We now define a K-functional

$$K^*_{\Delta, f}(f, t^{2\ell})_C = \inf_{g \in C^{2\ell}} (\|f - g\|_C + t^{2\ell} \|\Delta^\ell g\|_C), \tag{3.9}$$

where C is either $C(R^d)$ or $C(T^d)$. For this K-functional, we have the following direct result.

THEOREM 3.2. *For $f \in C(R^d)$ or $f \in C(T^d)$,*

$$\|B_{\ell, t} f - f\|_C \leqslant A K^*_{\Delta, \ell}(f, t^{2\ell})_C \tag{3.10}$$

and

$$\|S_{\ell, t} f - f\|_C \leqslant A K^*_{\Delta, \ell}(f, t^{2\ell})_C. \tag{3.11}$$

Remarks. In fact, for the direct result, (3.10) follows from (3.11). For the converse result, we use the estimate by $B_{\ell, t} f - f$ to obtain the (different) estimate by $S_{\ell, t} f - f$. We note the important special cases for $\ell = 1$:

$$\|B_t f - f\|_C \leqslant A \inf_{g \in C^2} (\|f - g\|_C + t^2 \|\Delta g\|_C) \tag{3.10$'$}$$

and

$$\|S_t f - f\|_C \leqslant A \inf_{g \in C^2} (\|f - g\|_C + t^2 \|\Delta g\|_C). \tag{3.11$'$}$$

Proof. We choose g^* such that $g^* \in C^{2\ell}$ and

$$\|f - g^*\|_C + t^{2\ell} \|\Delta^\ell g^*\|_C \leqslant 2K^*_{\Delta, \ell}(f, t^{2\ell})_C.$$

We estimate $(B_{\ell, t} - I)(f - g^*)$ and $(S_{\ell, t} - I)(f - g^*)$ using the boundedness of $B_{\ell, t} - I$ and of $S_{\ell, t} - I$ on C. We estimate $(B_{\ell, t} - I)\, g^*$ and $(S_{\ell, t} - I)\, g^*$ using (3.5) and (3.6). ∎

4. THE DIRECT RESULT FOR CLASSES OF NICE BANACH SPACES

We will prove the direct result (and in fact also the converse result) for two classes of Banach spaces given in the following definitions.

Definition 4.1. A Banach space B is called a homogeneous Banach space which we denote by $B \in$ H.B.S. if the following conditions are satisfied:

(I) $f \in B$ then f is a locally Lebesgue integrable function on R^d or T^d;

(II) $\|f(\cdot + a)\|_B = \|f(\cdot)\|_B$ (translation is an isometry);

(III) $\|f(\cdot + h) - f(\cdot)\|_B = o(1)$ $h \to 0$ (translation is strongly continuous).

Definition 4.2. A Banach space B is of class $\mathcal{N}$ which we denote by $B \in \mathcal{N}$ if the following conditions are satisfied:

(I) $f \in B$ then $f \in \mathcal{S}'$ where $\mathcal{S}'$ are the tempered distributions on R^d or T^d and B is continuously imbedded in $\mathcal{S}'$;

(II) $\|f(\cdot + a)\|_B = \|f(\cdot)\|_B$;

(III) $\mathcal{S}$, the Schwartz space of test functions, is dense in the Banach space X and $X^* \supset B$ (where X^* is the dual to X).

The spaces $L_p(R^d) \in$ H.B.S. for $1 \leqslant p < \infty$. $L_p(R^d) \in \mathcal{N}$ for $1 \leqslant p \leqslant \infty$, $C(R^d) \in$ H.B.S., and $C(R^d) \in \mathcal{N}$, Besov spaces ($1 \leqslant p < \infty$) belong to H.B.S., the space of measures and the dual to a Besov space belong to $\mathcal{N}$. The definitions of the K-functionals below will be different, but in case both apply, they will be shown in a later section to be equivalent. The conditions in Definitions 4.1 and 4.2 are standard and we just grouped them together under H.B.S. (which is standard) and $\mathcal{N}$ headings so that when theorems are stated, it is clear which conditions are used.

We define also the space B^r.

Definition 4.3. The space B^r is the space of functions in B whose first r strong derivatives (defined inductively) are also in B.

We can now state the main inequalities used for the direct result.

THEOREM 4.4. *Suppose* $f \in B$ *and* B *is a* H.B.S., *then*

(a) $f \in B^2$ *implies* (3.1) *and* (3.2);

(b) $f \in B^{2\ell+2}$ *implies* (3.3), (3.4) *and* (3.7);

(c) $f \in B^{2\ell}$ *implies* (3.5) *and* (3.6).

Proof. For $g \in B^*$ (B^* the dual to B), we define

$$F(x) = \langle f(x + \cdot), g(\cdot) \rangle.$$

($F(x)$ is not exactly convolution of f and g, it lost commutativity but gained marginally elsewhere.) The function $F(x)$ is continuous and bounded and, moreover, when $f \in B^r$ we have $F \in C^r$.

For $f \in B$, we have

$$B_t(F, x) = \langle B_t(f, x + \cdot), g(\cdot) \rangle$$

and

$$S_t(F, x) = \langle S_t(f, x + \cdot), g(\cdot) \rangle,$$

and hence similar relations are valid for $B_{\ell,t}F$ or $S_{\ell,t}F$ on one side with $B_{\ell,t}f$ or $S_{\ell,t}f$ on the other. For $f \in B^{2m}$, the relation

$$\Delta^m F(x) = \langle \Delta^m f(x + \cdot), g(\cdot) \rangle$$

holds. We now use Theorem 3.1 applied to $F(x)$ with $\|g\|_{B^*} = 1$ chosen appropriately, to obtain in a fairly standard way (see [Di, I; Di, II]) the result for $f \in B$. We give the method explicitly in the first of the seven cases treated. For any g such that $\|g\|_{B^*} = 1$ and $f \in B^2$, we have $F \in C^2$ and

$$\begin{aligned}
\|B_t F - F\|_C &\leqslant \frac{t^2}{2(d+2)} \|\Delta F\|_C \\
&\leqslant \frac{t^2}{2(d+2)} \sup_x |\langle \Delta f(x + \cdot), g(\cdot) \rangle| \\
&\leqslant \frac{t^2}{2(d+2)} \sup_x \|\Delta f(x + \cdot)\|_B \|g\|_{B^*} \\
&\leqslant \frac{t^2}{2(d+2)} \|\Delta f\|_B.
\end{aligned}$$

On the other hand,

$$\begin{aligned}\|B_t f - f\|_B &= \sup_{\|g\|_{B^*}=1} |\langle B_t(f,\cdot) - f(\cdot), g(\cdot)\rangle| \\ &\leqslant \sup_{\|g\|_{B^*}=1} \sup_x |\langle B_t(f, x+\cdot) - f(x+\cdot), g(\cdot)\rangle| \\ &= \sup_{\|g\|_{B^*}=1} \|B_t F - F\|_C.\end{aligned}$$

Similarly, one can prove the other inequalities. ∎

We now define the K-functionals which we need by

$$K^*_{\Delta,\ell}(f, t^{2\ell})_B = \inf_{g\in B^{2\ell}} (\|f-g\|_B + t^{2\ell}\,\|\Delta^\ell g\|_B). \tag{4.1}$$

That is, (1.1) with $B^{2\ell} = J(\ell)$. From this, we deduce the direct result.

THEOREM 4.5. *For $f \in B$ and $B \in$ H.B.S., we have*

$$\|B_{\ell,t} f - f\|_B \leqslant C K^*_{\Delta,\ell}(f, t^{2\ell})_B \tag{4.2}$$

and

$$\|S_{\ell,t} f - f\|_B \leqslant C K^*_{\Delta,\ell}(f, t^{2\ell})_B. \tag{4.3}$$

It is, perhaps, worth mentioning that in the basic special case when $\ell = 1$, the above yields

$$\|B_t f - f\|_B \leqslant C \inf_{g\in B^2} (\|f-g\|_B + t^2\,\|\Delta g\|_B) \equiv C K^*_\Delta(f, t^2)_B$$

and

$$\|S_t f - f\|_B \leqslant C K^*_\Delta(f, t^2)_B.$$

Proof. We note that the technique of the proof of Theorem 4.4 implies also that B_t and S_t are contractions on B. Therefore,

$$\|B_{\ell,t} f\|_B \leqslant \left(\frac{2^{2\ell}}{\binom{2\ell}{\ell}} - 1\right) \|f\|_B \qquad \text{and} \qquad \|S_{\ell,t} f\|_B \leqslant \left(\frac{2^{2\ell}}{\binom{2\ell}{\ell}} - 1\right) \|f\|_B.$$

We now follow the proof of Theorem 3.2, choosing $g^* \in B^{2\ell}$ such that

$$\|f - g^*\|_B + t^{2\ell}\,\|\Delta^\ell g^*\|_B \leqslant 2K^*_{\Delta,\ell}(f, t^{2\ell})_B,$$

and estimate both $B_{\ell,t}-I$ and $S_{\ell,t}-I$ on $f-g^*$ using boundedness, and on g^* using (4.2) and (4.3). ∎

To obtain our results for spaces of type $\mathcal{N}$, we note that

$$\langle f(\cdot+h)-f(\cdot), \psi(\cdot)\rangle = o(1), \qquad h\to 0 \quad \forall f\in\mathcal{S}', \forall\psi\in\mathcal{S}. \tag{4.4}$$

We define

$$F(x)=\langle f(x+\cdot), \psi(\cdot)\rangle = f(\tau_x\psi), \tag{4.5}$$

where

$$\tau_x\psi(u)=\psi(u-x) \tag{4.6}$$

and note, following the standard argument [St-We, p. 23], that $F(x)$ is a C^∞ function. Moreover, for $f\in B$,

$$|F(x)|\leqslant \|F\|_C \leqslant \|f\|_B\,\|\psi\|_X. \tag{4.7}$$

The average B_tf and S_tf were defined for locally Lebesgue integrable functions. We can define them on $\mathcal{S}'$ by

$$\langle B_tf, \psi\rangle = \langle f, B_t\psi\rangle, \qquad \langle S_tf, \psi\rangle = \langle f, S_t\psi\rangle, \tag{4.8}$$

for all $\psi\in\mathcal{S}$ (4.8) extends the definitions of (1.2) and (1.3) because of the symmetry of B_t and S_t.

Moreover, since $B\subset X^*$, or elements of B are functionals on X, and S is dense in X, B_tf and S_tf are elements of X^* As B_t or S_t commute with τ_x, we can obtain

$$B_t(F,x)=\langle B_t(f, x+\cdot), \psi(\cdot)\rangle \tag{4.9}$$

and

$$S_t(F,x)=\langle S_t(f, x+\cdot), \psi(\cdot)\rangle, \tag{4.10}$$

and hence the density of $\mathcal{S}$ in X implies that B_t and S_t are contractions in X^*.

The operator Δ^m on $\mathcal{S}'$ is defined as usual by

$$\langle \Delta^m f, \psi\rangle = \langle f, \Delta^m\psi\rangle \qquad \forall\psi\in\mathcal{S}. \tag{4.11}$$

This implies

$$\Delta^m F(x)=\langle \Delta^m f(x+\cdot), \psi(\cdot)\rangle. \tag{4.12}$$

We can now state the basic inequalities for $B\in\mathcal{N}$.

THEOREM 4.6. *Suppose* $f \in B \subset \mathcal{S}'$ *and* $B \in \mathcal{N}$ *with the space* X *of Definition* 4.3 *satisfying* $X^* \supset B$. *Then, with* $B_t f$ *and* $S_t f$ *defined by* (4.8) *and* $\Delta^\ell f$ *by* (4.11), *we have*:

(a) $\Delta f \in X^*$ *implies* (2.1) *and* (2.2) *with the norm* X^*;

(b) $\Delta^{\ell+1} f \in X^*$ *implies* (2.3), (2.4) *and* (2.7) *with the norm of* X^*; *and*

(c) $\Delta^\ell f \in X^*$ *implies* (2.5) *and* (2.6) *with the norm of* X^*.

Remark 4.7. Because of the construction above, if $B_t f$ or $S_t f$ or $\Delta^\ell f$ are in B, the appropriate norm $\| \ \|_{X^*}$ can be replaced by $\| \ \|_B$. An interesting possible situation is when $B_t f$ is in B and Δf only in X^*. In this case, we obtain $\|B_t f - f\|_B \leqslant (t^2/2(d+2)) \, \|\Delta f\|_{X^*}$. This situation occurs, for instance, when $B = L_1(T^d)$, $X = C(T^d)$ and $X^* = \mathcal{M}$, where $\mathcal{M}$ is the space of measures. (In this case, $L_1(T^d)$ is imbedded in $\mathcal{M}$ in the natural way.) The inequality

$$\|B_t f - f\|_{L_1(T^d)} \leqslant \frac{t^2}{2(d+2)} \, \|\Delta f\|_{\mathcal{M}}$$

is valid, makes sense and is the crucial descriptive direct direction of the saturation class, i. e., the class of functions for which $\|B_t f - f\|_{L_1(T^d)} = O(t^2)$ $t \to 0+$.

Proof. To prove our theorem, we observe that, as in the proof of Theorem 4.4, we have one method that fits all cases. We prove the validity of (2.1) with the norm X^*, and other parts follow in a similar fashion. We apply (2.1) with the C norm proved in Theorem 3.1 to $F(x)$ defined by (4.5). The conditions apply, as F is bounded if $f \in B$ and $\psi \in S \subset X$, $F \in C^\infty$ locally, and $\Delta F \in C$ as

$$|\Delta F(x)| = |\langle \Delta f(x + \cdot), \psi \rangle| \leqslant \|\Delta f\|_{X^*} \, \|\psi\|_X.$$

We now write

$$\begin{aligned} \|B_t f - f\|_{X^*} &= \sup \{ |\langle B_t f - f, \psi \rangle|; \ \|\psi\|_X = 1, \ \psi \in S \} \\ &\leqslant \frac{t^2}{2(d+2)} \, \|\Delta F(\cdot)\|_C \\ &\leqslant \frac{t^2}{2(d+2)} \, \|\Delta f\|_{X^*}. \quad \blacksquare \end{aligned}$$

For the measure of smoothness, we define the appropriate K-functional for Banach space B where $B \in \mathcal{N}$ by

$$K_{\ell, \Delta}(f, t^{2\ell})_B = \inf_{g,\, \Delta^\ell g \in X^*} (\|f - g\|_{X^*} + t^{2\ell} \, \|\Delta^\ell g\|_{X^*}). \tag{4.13}$$

Note that $\Delta^\ell g$ is given in the sense of (4.11).

We will show later that, for spaces B for which the two K-functionals $K_{\ell, \Delta}$ and $K^*_{\ell, \Delta}$ were given, they yield essentially the same concept. We can now state and prove the direct result for spaces B, $B \in \mathcal{N}$.

THEOREM 4.8. *For $f \in B \subset S'$, $B \in \mathcal{N}$, we have*

$$\|B_{\ell, t} f - f\|_{X^*} \leqslant C K_{\ell, \Delta}(f, t^{2\ell})_B$$

and

$$\|S_{\ell, t} f - f\|_{X^*} \leqslant C K_{\ell, \Delta}(f, t^{2\ell})_B.$$

When $B_t f \in B$, then $B_{\ell, t} f \in B$ and $\|B_{\ell, t} f - f\|_B = \|B_{\ell, t} f - f\|_{X^*}$, and when $S_t f \in B$, then $S_{\ell, t} f \in B$ and $\|S_{\ell, t} f - f\|_B = \|S_{\ell, t} f - f\|_{X^*}$.

Proof. The proof follows that of Theorem 4.5 using Theorem 4.6 here rather then Theorem 4.4. ∎

5. CONVERSE INEQUALITIES OF TYPES B AND D

In this section, we obtain strong converse inequalities of type B (in the terminology of [Di-Iv]) for the approximation process $B_{\ell, t} f - f$. That is, we approximate the appropriate K-functional by two terms, $\|B_{\ell, t} f - f\|$ and $\|B_{\ell, t\rho} f - f\|$ with some ρ. In the next section, we will show, for $\ell = 1$ and all d, that $\|B_t f - f\|$ is sufficient (that is, $\rho = 1$). From the strong converse inequality of type B for $B_{\ell, t} f - f$, we will deduce a strong converse inequality of type D for $S_{\ell, t} f - f$, that is, an estimate of the K-functional by $\sup_{0 < r \leqslant t} \|S_{\ell, \tau} f - f\|$. We cannot strive for an estimate of the K-functional by $S_{\ell, t} f - f$ in a strong converse inequality of type B (or A), as it is well-known not to be valid for $\ell = 1$, $d = 1$, and $B = L_p$, $1 \leqslant p \leqslant \infty$.

As is common, a Bernstein type inequality is crucial for the converse result. Using the methods described in details in the last section, it is evident that the appropriate Bernstein inequality can be proved for the space C or even just locally in that space and then copied to the general situation.

THEOREM 5.1. *For $f \in C(R^d)$ or $f \in C(T^d)$ and ξ any direction in R^d, we have*

$$\left|\frac{\partial}{\partial \xi} B_t(f, x)\right| \leqslant \frac{d}{t} \int_{St} |f(x+v)|\, d\sigma(v). \tag{5.1}$$

Proof. We rewrite $B_t(f, x)$ using integration in the ξ direction first and then on $B(\xi) = \{u\colon u \cdot \xi = 0,\ |u| \leqslant 1\}$. To compute $(\partial/\partial\xi)\, B_t(f, x)$, we also use the one-dimensional classical identity

$$\frac{d}{dx} \int_{-a}^{a} f(x+v)\, dv = f(x+a) - f(x-a)$$

(which is clearly valid here). Hence, we obtain

$$\begin{aligned}\frac{\partial}{\partial \xi} B_t(f, x) &= \frac{1}{m(B)\, t^d} \frac{\partial}{\partial \xi} \int_{B(\xi)\, t} \int_{-\sqrt{t^2-|v|^2}}^{\sqrt{t^2-|v|^2}} f(x+v+y\xi)\, dy\, dv \\ &= \frac{1}{m(B)\, t^d} \int_{B(\xi)\, t} \{f(x+v+\sqrt{t^2-|v|^2}\, \xi) \\ &\qquad - f(x+v-\sqrt{t^2-|v|^2}\, \xi)\}\, dv.\end{aligned}$$

Therefore,

$$\begin{aligned}\left|\frac{\partial}{\partial \xi} B_t(f, x)\right| \leqslant \frac{1}{m(B)\, t^d} \int_{B(\xi)\, t} &\{|f(x+v+\sqrt{t^2-|v|^2}\, \xi)| \\ &+ |f(x+v-\sqrt{t^2-|v|^2}\, \xi)|\}\, dv.\end{aligned} \tag{5.2}$$

We note now that $d\sigma(v)$ is bigger than its projection on $E(\xi) = \{v;\ v \cdot \xi = 0\}$ to obtain (5.1) from (5.2). ∎

From Theorem 5.1, we deduce the following corollary.

COROLLARY 5.2. *For $f \in L_p(R^d)$ or $f \in L_p(T^d)$, $1 \leqslant p < \infty$,*

$$\|\text{grad}(B_t f)\|_{L_p} \leqslant \frac{d}{t} \|f\|_{L_p}. \tag{5.3}$$

For $f \in C(R^d)$ or $f \in C(T^d)$,

$$\|\text{grad}(B_t f)\|_C \leqslant \left(\frac{2m(B_{d-1})}{m(B_d)}\right) t^{-1} \|f\|_C, \tag{5.4}$$

where $m(B_\ell)$ is the volume of the ℓ-dimensional unit ball.

Proof. Using (5.2) and observing that if $f \in C$ so does $(\partial/\partial\xi)\, B_t(f, x)$, and hence grad $B_t(f, x)$, grad $B_t(f, x)$ achieves maximum in $C(T^d)$ at $x_0 \in T^d$, and is close to supremum at a point $x_0 \in R^d$ for $f \in C(R^d)$. At that point, we estimate $(\partial/\partial\xi)\, B_t(f, x_0)$. We now use (5.2) again to obtain

$$\left|\frac{\partial}{\partial\xi} B_t(f, x_0)\right| \leqslant \frac{2m(B_{d-1})}{m(B_d)\, t} \|f\|_C,$$

where $m(B_\ell)$ is the measure of the ℓ-dimensional unit ball. We now use (5.1) for $f \in C$ and obtain

$$|\text{grad } B_t(f, x)| \leqslant \frac{d}{t} \int_{St} |f(x+v)|\, d\sigma(v)$$

and hence, for $f \in C$,

$$\|\text{grad } B_t(f, x)\|_{L_p} \leqslant \frac{d}{t} \|f\|_{L_p}$$

which implies (5.3) as C is dense in L_p, $1 \leqslant p < \infty$. ∎

Actually, (5.1) is valid for L_∞ as well, and hence

$$\|\text{grad } B_t f\|_\infty \leqslant \frac{d^{3/2}}{t} \|f\|_\infty$$

which is somewhat worse than (5.3) and certainly worse than (5.4), but is of no consequence. In fact, the method of Theorem 5.6 below implies (5.3) for L_∞ as well.

We now obtain the following commutativity result that is more an observation than a hard earned proven theorem.

THEOREM 5.3. *Suppose, f has locally continuous first derivatives. Then*

$$\frac{\partial}{\partial\xi} B_t(f, x) = B_t\left(\frac{\partial}{\partial\xi} f, x\right) \tag{5.5}$$

and

$$\text{grad } B_t(f, x) = B_t(\text{grad } g, x). \tag{5.6}$$

From these results, we deduce the following Bernstein estimate.

THEOREM 5.4. *For $f \in C(R^d)$ or $f \in C(T^d)$, we have*

$$\|\Delta B_t B_\tau f\|_C \leqslant \frac{d^2}{t\tau} \|f\|_C. \tag{5.7}$$

Proof. We first assume that $f \in C^1$, we then choose $g \in L^1$ so that $\|g\|_1 = 1$ and

$$\langle g, \Delta B_t B_\tau f \rangle \geqslant \|\Delta B_t B_\tau f\|_C - \varepsilon.$$

As $f \in C^1$, $B_\tau f \in C^2$ and hence, using Theorem 5.3, we have

$$\begin{aligned}\langle g, \Delta B_t B_\tau f \rangle &= \langle g, B_t \Delta B_\tau f \rangle \\ &= \langle B_t g, \Delta B_\tau f \rangle = \langle \text{grad } B_t g, \text{grad } B_\tau f \rangle.\end{aligned}$$

Therefore

$$\begin{aligned}|\langle g, \Delta B_t B_\tau f \rangle| &\leqslant \|\text{grad } B_t g\|_1 \, \|\text{grad } B_\tau f\|_C \\ &\leqslant \frac{d^2}{t\tau} \|g\|_1 \, \|f\|_C = \frac{d^2}{t\tau} \|f\|_C.\end{aligned}$$

Since ε is arbitrary, the result is valid for $f \in C^1$, and since C^1 is dense in C (with the C norm), (5.7) holds. ∎

In fact, the estimate is somewhat better as $2m(B_{d-1})/m(B_d)$ is smaller than d and is asymptotically $\sim Cd^{1/2}$ (with different C for odd and even d). This would improve some estimates, but would not save us from the discussion in Section 6.

As a corollary of the above results, we obtain the following result also.

THEOREM 5.5. *For $f \in C(R^d)$ or $f \in C(T^d)$, we have*

$$\|\Delta^\ell B_{t_1} \cdots B_{t_{2\ell}} f\|_C \leqslant \frac{d^{2\ell}}{t_1 \cdots t_{2\ell}} \|f\|_C. \tag{5.8}$$

Proof. To use earlier theorems, we just have to note that for $g \in C^{2r}$ ($g \in C^{2r}_{loc}$ is sufficient), $\Delta^r B_t g = B_t \Delta^r g$ and use Theorem 5.4 repeatedly. ∎

From these results, we obtain, using the notation and proof in the last section, the following theorem.

THEOREM 5.6. *For $f \in B$, $B \in$ H.B.S., we have* (5.1), (5.2), (5.7), *and* (5.8) *with the norm B replacing C. For $f \in B$, $B \in \mathcal{N}$, we have* (5.1), (5.2), (5.7), *and* (5.8) *with the X^* norm, where X^* is given in Definition* 4.2, *replacing the norm of C.*

From the above theorems, we deduce the strong converse inequality of type B for $B_{\ell,t}f-f$, and from it the strong converse inequality of type D for $S_{\ell,t}f-f$. We state the result simultaneously, which is somewhat more economical and saves us unnecessary repetition.

THEOREM 5.7. *Suppose $f\in B$. Suppose also that $B_{\ell,t}f$ and $S_{\ell,t}f$ are given by* (2.6) *and* (2.7), *respectively, where $B_t f$ and $S_t f$ are given by* (1.2), (1.3), *and* (4.8). *Then*:

(a) *for $B\in$ H.B.S. and $K^*_{\ell,\Delta}(f,t^{2\ell})_B$ is given by* (4.1), *we have*

$$K^*_{\ell,\Delta}(f,t^{2\ell})_B\leqslant C(\|B_{\ell,t}f-f\|_B+\|B_{\ell,t\rho}f-f\|_B)$$

and

$$K^*_{\ell,\Delta}(f,t^{2\ell})_B\leqslant C\sup_{0<\tau\leqslant t}\|S_{\ell,\tau}f-f\|_B;$$

(b) *for $B\in\mathcal{N}$, $K_{\ell,\Delta}(f,t^{2\ell})_B$ given by* (4.13) *and X^* given in Definition* 4.2, *we have*

$$K_{\ell,\Delta}(f,t^{2\ell})_B\leqslant C(\|B_{\ell,t}f-f\|_{X^*}+\|B_{\ell,t\rho}f-f\|_{X^*})$$

and

$$K_{\ell,\Delta}(f,t^{2\ell})_B\leqslant C\sup_{0<\tau\leqslant t}\|S_{\ell,\tau}f-f\|_{X^*}.$$

In both (a) *and* (b), *$C=C(\ell,d)$ and $\rho\leqslant\rho_0=\rho_0(\ell,d)$ are independent of f and B.*

Proof. We first show that the strong converse inequality of type D for $S_{\ell,t}f-f$ follows from the strong converse inequality of type B for $B_{\ell,t}f-f$. This is so, as

$$K^*_{\ell,\Delta}(f,t^{2\ell})_B\leqslant C\sup_{0<\eta\leqslant t}\|B_{\ell,\eta}f-f\|_B$$

(recall $\rho\leqslant\rho_0$). Then we observe that

$$\begin{aligned}\sup_{\eta\leqslant t}\|B_{\ell,\eta}f-f\|_B&\leqslant\sup_{\eta\leqslant t}\left\|\frac{1}{d}\int_0^\eta(S_{\ell,\tau}f-f)\,\tau^{d-1}\,d\tau\right\|_B\\&\leqslant\sup_{\tau\leqslant t}\|S_{\ell,\tau}f-f\|_B.\end{aligned}$$

When the estimate is given for $B\in\mathcal{N}$, we use the same argument but the operators are on $\psi\in\mathcal{S}$ in the norm X, and this is done in the way that repeats earlier arguments.

To obtain the strong converse of type B, we define

$$I_{\ell,t}(f)=\|f-B_{\ell,t}^{2\ell+2}f\|+t^{2\ell}\,\|\Delta^{\ell}B_{\ell,t}^{2\ell+2}f\|, \tag{5.9}$$

where the norm is in B or X^* as appropriate. It is clear, because of (5.8) and Theorem 5.6, that $I_{\ell,t}(f)$ is bigger than the K-functional in question. As $\|B_{\ell,t}f\|\leqslant C_\ell\,\|f\|$, we have

$$\begin{aligned}\|f-B_{\ell,t}^{2\ell+2}f\| &\leqslant \sum_{j=0}^{2\ell+1}\|B_{\ell,t}^{j}(f-B_{\ell,t}f)\|\\ &\leqslant C\,\|f-B_{\ell,t}f\|.\end{aligned}$$

To estimate $t^{2\ell}\,\|\Delta^{\ell}B_{\ell,t}^{2\ell+2}f\|$, we use

$$\begin{aligned}&\|B_{\ell,t\rho}B_{\ell,t}^{2\ell+2}f-B_{\ell,t}^{2\ell+2}f-A(\ell,d)(t\rho)^{2\ell}\,\Delta^{\ell}B_{\ell,t}^{2\ell+2}f\|\\ &\quad=C_*(\ell,d)(t\rho)^{2\ell+2}\,\|\Delta^{\ell+1}B_{\ell,t}^{2\ell+2}f\|\end{aligned} \tag{5.10}$$

which is (3.7) with $B_{\ell,\rho t}f$ taking the place of $B_{\ell,t}f$, and $B_{\ell,t}^{2\ell+2}f$ taking the place of f there. We observe that (5.10) and Theorem 5.6 imply that the conditions of Theorems 4.4 and 4.6 (for (3.7)) are applicable to $B_{\ell,t}^{2\ell+2}f$.

We utilize (5.9) and Theorem 5.6, operated on $B_{mt}B_{kt}$ with $1\leqslant k,\ m\leqslant\ell$, and commutativity of Δ, B_η, and B_θ to obtain

$$\begin{aligned}C_*(\ell,d)(t\rho)^{2\ell+2}\,|\Delta^{\ell+1}B_{\ell,t}^{2\ell+2}f| &\leqslant C_*(\ell,d)(d\tau)^{2\ell+2}\,\|\Delta B_{\ell t}^{2}\Delta^{\ell}b_{\ell,t}^{2\ell}f\|\\ &\leqslant C_*(\ell,d)(t\rho)^{2\ell}\,\rho^2C\,\|\Delta^{\ell}B_{\ell,t}^{2\ell}f\|\end{aligned}$$

with C independent of t and f.

We now choose ρ, so that

$$C_*(\ell,d)\,\rho^2C\leqslant\frac{1}{2}\,|A(\ell,d)|\equiv\frac{1}{2^{\ell+1}}\,\frac{\ell\,!}{(d+2)\cdots(d+2\ell)}. \tag{5.11}$$

We estimate

$$\begin{aligned}\|\Delta^{\ell}B_{\ell,t}^{2\ell}f\| &\leqslant \|\Delta^{\ell}B_{\ell,t}^{2\ell+2}f\|+\|\Delta^{\ell}B_{\ell,t}^{2\ell}(f-B_{\ell,t}^{2}f)\|\\ &\leqslant \|\Delta^{\ell}B_{\ell,t}^{2\ell+2}f\|+A_*t^{-2\ell}\,\|f-B_{\ell,t}f\|.\end{aligned}$$

Combining the above, we obtain

$$t^{2\ell}\,\|\Delta^{\ell}B_{\ell,t}^{2\ell+2}f\|\leqslant M(\rho)(\|f-B_{\ell,\rho t}f\|+\|f-B_{\ell,t}f\|)$$

for any ρ chosen to satisfy (5.11). ∎

In fact, $I(\ell, t)$ was shown to be the realization of the K-functionals $K^*_{\ell,\Delta}(f, t^{2\ell})_B$ and $K_{\ell,\Delta}(f, t^{2\ell})_B$ for $B \in$ H.B.S. and $B \in \mathcal{N}$, that is, $I(\ell, t)$ was shown to be equivalent to the K-functional. Therefore, if $f \in B$ and $B \in$ H.B.S. and $B \in \mathcal{N}$, we have

$$K_{\ell,\Delta}(f, t^{2\ell})_B \approx K^*_{\ell,\Delta}(f, t^{2\ell})_B.$$

Changing somewhat the structure of the realization, we can have equivalent with the K-functional defined by (1.1) with different $J(\ell)$ (see (1.1)).

6. THE STRONG-CONVERSE INEQUALITY OF TYPE A

In Theorem 5.7, the estimate of the K-functional by the rate of approximation involved two terms (if $\rho_0 < 1$ there). That is, we achieved, in the terminology of [Di-Iv], a strong converse inequality (S.C.I.) of type B. In many cases, one term is sufficient and in such cases we have a S.C.I. of type A. While S.C.I. of type B are sufficient for many purposes (establishing "realization" and implying all classical converse inequalities, for example), some serious efforts were made in special cases to prove the more elegant S.C.I. of type A even when type B was available and relatively easy (see [Di-Iv, Sect. 4; To; Kn-Zh]). In [Di-Iv, Sect. 4], an easy proof is shown to follow a difficult estimate on constants, in [Kn-Zh], a method to estimate the constants in case the operators are positive (with some additional restrictions) is given, and in [To], an intricate modification of the "parabola" technique is used for positive operators with the C norm. While the problem of proving a S.C.I. of type A for $B_{\ell,t}f - f$, for general ℓ, even in case $d = 1$, remains open, we establish such a result for $\ell = 1$ and all d.

THEOREM 6.1. *Suppose* $f \in B$, $B \in$ H.B.S., $B_t f$ *given by* (1.2), *and* $K^*_{\ell,\Delta}(f, t^{2\ell})_B$ *given by* (4.1). *Then*

$$K^*_\Delta(f, t^2)_B \equiv K^*_{1,\Delta}(f, t^2)_B \approx \|B_t f - f\|_B.$$

Suppose $f \in B$, $B \in \mathcal{N}$, $B_t f$ *given by* (1.2) *and* (4.8), X^* *given in Definition* 4.2 *and* $K_{\ell,\Delta}(f, t^{2\ell})_B$ *given by* (4.13). *Then*

$$K_\Delta(f, t^2)_B \equiv K_{1,\Delta}(f, t^2)_B \approx \|B_t f - f\|_{X^*}.$$

Proof. Following the steps and results of the last two sections, it is sufficient to prove our result for $B = C(C(R^d)$ or $C(T^d))$. We now write

$$I_t(f) = \|f - B_t^{m+4} f\| + t^2 \|\Delta B_t^{m+4} f\| \tag{6.1}$$

and note, following the proof of Theorem 5.7, that it is sufficient to show

$$t^2 \|\Delta B_t^{m+4} f\| \leqslant C \|B_t f - f\|$$

for some integer m. Using (3.7) for $\ell = 1$, we have

$$\left\| B_t g - g + \frac{t^2}{2(d+2)} \Delta g \right\| \leqslant \frac{t^4}{8(d+2)(d+4)} \|\Delta^2 g\| \tag{6.2}$$

which we apply to $g = B_t^{m+4} f$. We now use (5.5) with $\tau = t$, and obtain our theorem following the last section for $d^2/4(d+4) < 1$ or $d < 7$. We note here that, for $1 \leqslant d \leqslant 6$ the above simple careful application of the method in [Di-Iv] implies our result. For higher dimensions, it is sufficient to prove, using the above mentioned technique, that for every $\varepsilon > 0$ there exists $m = m(\varepsilon)$ such that

$$\|\Delta B_t^m g\| \leqslant \varepsilon t^2 \|g\|. \tag{6.3}$$

Once a result like (6.3) (in fact, $\varepsilon < 4(d+4)$ is sufficient) is proved, a combination of (6.2), (6.3) and

$$\|B_t^k f - f\| \leqslant C \|B_t f - f\|$$

completes the proof. Hence, we proved our result for all d, pending the proof of (6.3) which is sufficient to show for the case B is the space $C(R^d)$ or $C(T^d)$ and is given a lemma below. ▮

The proof of (6.3) follows an ingeneous method in [Kn-Zh] which is developed for proving such inequalities. (In [Kn-Zh], they also give a repetition of estimates in [Di-Iv] with a change of name from S.C.I. to lower estimate.) The conditions set in [Kn-Zh] are not exactly fitting for our case, however the ideas given in [Kn-Zh] are used here.

LEMMA 6.2. *For $f \in C(R^d)$ or $f \in C(T^d)$ and $\varepsilon > 0$, there exists $r = r(\varepsilon)$ such that*

$$\|\Delta B_t^{6r} f\|_C \leqslant \varepsilon t^{-2} \|f\|_C. \tag{6.4}$$

Proof. It is sufficient to find r so that

$$\left\| \frac{\partial}{\partial \xi} B_t^{3r} f \right\|_C \leqslant \sqrt{\frac{\varepsilon}{d}}\, t^{-1} \|f\|_C \tag{6.5}$$

as (6.5) will imply, using (5.3),

$$\left\| \left(\frac{\partial}{\partial \xi} \right)^2 B_t^{6r} f \right\|_C \leqslant \frac{\varepsilon}{d} t^{-2} \|f\|_C$$

from which (6.4) is immediate. We denote $\varepsilon_1 = \sqrt{\varepsilon/d}$. The derivative of the function $\chi(Bt)$ which is the kernel of $B_t f$ is not a function, so to square it and then divide by it, following [Kn-Zh], is not possible. We now consider $B_t(f, x)$ like the convolution

$$B_t(f, x) = \frac{1}{m(B)\, t^d} \int_{Bt} f(x-u)\, du$$

with the kernel $(1/m(B)\, t^d)\, \chi(Bt)$. For $t = 1$, we have

$$B_1^3(f, x) = \int f(x-u)\, \varphi(u)\, du,$$

where $\varphi(u) \in L_\infty^{(2)}$, that is $(\partial^2/\partial\xi\, \partial\eta)\, \varphi(\cdot) \in L_\infty$ for all ξ and η, $\varphi(u)$ is radially symmetric and supp $\varphi(u) = \{u\colon |u| \leqslant 3\}$. Moreover,

$$\begin{aligned} B_t^3(f, x) &\equiv \int f(x-u)\, \varphi_t(u)\, du \\ &\equiv \int f(x-u)\, \varphi\left(\frac{u}{t}\right) t^{-d} du. \end{aligned} \tag{6.6}$$

In other words, supp $\varphi_t(u) = \{u\colon |u| \leqslant 3t\}$. We have $\varphi_t(u) \geqslant 0$, as φ_t is a convolution of positive functions. We also have that

$$\int \varphi_t(x)\, dx = 1 \qquad \text{and} \qquad \int \frac{\partial}{\partial \xi}\, \varphi_t(x)\, dx = 0$$

as a result of $B_t 1 = 1$. As $\varphi(u) > 0$ for $\{u\colon |u| < 3\}$, $\varphi(u) = 0$ for $\{u\colon |u| \geqslant 3\}$, and $\varphi(x)$ has a zero of order 2 at $\{u\colon |u| = 3\}$, we have

$$\int \left(\frac{\partial}{\partial \xi}\, \varphi(x) \right)^2 \varphi(x)^{-1}\, dx = C > 0.$$

Substitution implies now

$$\int \left(\frac{\partial}{\partial \xi}\, \varphi_t(x) \right)^2 \varphi_t(x)^{-1}\, dx = C t^{-2}. \tag{6.7}$$

Hence, while the technique of [Kn-Zh] does not work directly for $B_t f$, it does work for $B_t^3 f$. We now outline the proof (6.5). We write

$$\frac{\partial}{\partial\xi} B_t^{3r}(f, x_0) = \frac{1}{r} \int \cdots \int \left\{ \sum_{k=0}^{r-1} \frac{(\partial/\partial\xi)\, \varphi_t(x_k - x_{k+1})}{\varphi_t(x_k - x_{k+1})} \right.$$

$$\left. \times\, \varphi_t(x_0 - x_1) \cdots \varphi_t(x_{r-1} - x_r) \right\} f(x_r)\, dx_1 \cdots dx_r.$$

We observe that, in spite of the fact that $((\partial/\partial\xi)\, \varphi_t(x))/\varphi_t(x)$ is not integrable, the integral above converges. We now use the Cauchy–Schwartz inequality and obtain

$$\left| \frac{\partial}{\partial\xi} B_t^{3r}(f, x_0) \right|^2 \leqslant \left\{ \frac{1}{r^2} \int \cdots \int \left(\sum_{k=0}^{r-1} \frac{(\partial/\partial\xi)\, \varphi_t(x_k - x_{k+1})}{\varphi_t(x_k - x_{k+1})} \right)^2 \right.$$

$$\left. \times\, \Phi_t(x_0, ..., x_r)\, dx_1 \cdots dx_r \right\}$$

$$\times \left\{ \int \cdots \int \Phi_t(x_0, ..., x_r)\, |f(x_r)|^2\, dx_1 \cdots dx_r \right\}$$

$$\equiv I \times J,$$

where $\Phi_t(x_0, ..., x_r) = \varphi_t(x_0 - x_1) \cdots \varphi_t(x_{r-1} - x_r)$ and dx_i indicates a d-dimensional integration as x_i is d-dimensional and $d/d\xi$ is differentiation in ξ direction of the x_k variable. From properties of $\varphi_t(x)$, we have $\Phi_t(x_0, ..., x_r) \geqslant 0$ and

$$\int \cdots \int \Phi_t(x_0, ..., x_r)\, dx_1 \cdots dx_r = 1.$$

Hence,

$$J \leqslant \|f\|_C^2.$$

To estimate I, we note that for $1 \leqslant k+1 \leqslant r$

$$\int \cdots \int \left(\frac{\partial}{\partial\xi} \varphi_t(x_k - x_{k+1}) \right)^2 (\varphi(x_k - x_{k+1}))^{-2}\, \Phi_t(x_0, x_1 \cdots x_r)\, dx_1 \cdots dx_r$$

$$= \int \left(\frac{\partial}{\partial\xi} \varphi_t(x_k - x_{k+1}) \right)^2 (\varphi_t(x_k - x_{k+1}))^{-1}\, dx_{k+1} = Ct^{-2}.$$

We further observe that for $0 \leqslant k, \ell < r$ and $k \neq \ell$ we have

$$\int \cdots \int \frac{(\partial/\partial\xi)\, \varphi_t(x_k - x_{k+1})\, (\partial/\partial\xi)\, \varphi_t(x_\ell - x_{\ell+1})}{\varphi_t(x_k - x_{k+1})\, \varphi_t(x_\ell - x_{\ell+1})} \times \Phi_t(x_0, ..., x_r)\, dx_1 \cdots dx_r = 0,$$

where $(\partial/\partial\xi)\, \varphi_t(x_m - x_{m+1})$ is differentiation in the ξ direction with respect to the x_m variable.

Therefore,

$$I \leqslant \frac{C}{r^2}\, t^{-2}$$

and (6.5) follows for the choice $C/r^2 \leqslant \varepsilon/d$ which implies (6.4). ∎

7. CONCLUSIONS AND COMPARISONS

It is clear from arguments in Section 5 that

$$\|f - B_{m,t}^{2k+2} f\| + t^{2\ell}\, \|\Delta^\ell B_{m,t}^{2k+2} f\|$$

with the B norm or X^* norm and $k, m \geqslant \ell$ will also form a "realization" of the K-functional., i.e., will be equivalent to it. This can be used to give the usual relations between $K_{\ell,\Delta}(f, t^{2\ell})_B$ for different ℓ for a Banach space over T^d, the above was done in [Ch-Di] using a different realization. The technique of using realization for comparing K-functionals for different ℓ is the same.

The realization will also yield a comparison with the classical K-functional ($B \in$ H.B.S.) given by

$$K_{2\ell}^*(f, t^{2\ell})_B = \inf_{g \in B^{2\ell}} \left(\|f - g\|_B + t^{2\ell} \sup_{|\xi| = 1} \left\| \frac{\partial^{2\ell}}{\partial\xi^{2\ell}}\, g \right\|_B \right). \tag{7.1}$$

For this K-functional, we have

$$\begin{aligned} K_{\ell,\Delta}^*(f, t^{2\ell})_B &\leqslant C K_{2\ell}^*(f, t^{2\ell})_B \\ &\leqslant C^2 K_{\ell+1,\Delta}^*(f, t^{2\ell+2})_B \end{aligned} \tag{7.2}$$

and a similar result can be achieved for $B \in \mathcal{N}$. For L_p, $1 < p < \infty$, we have

$$K_{\ell,\Delta}^*(f, t^{2\ell})_p \approx K_{2\ell}^*(f, t^{2\ell})_p.$$

In Section 6, a strong converse inequality of type A is discussed for $B_t f - f$. For $B_{\ell,t} f - f$, such a result can be shown following [Di-Iv], when $d=1$ and small ℓ, that the same is true. For higher dimensions and even for $d=1$ and higher ℓ, it is open whether a S.C.I. of type A is valid. We conjecture that it is, but our proof does not yield such a result. It should be noted that, as $B_{\ell,t}$ is not a positive operator, the methods used in [To] and in [Kn-Zh] do not apply.

Using [Di-Iv, Sect. 10], one can use iterations rather than combinations for higher degrees of smoothness. This would yield, under the condition of Theorem 6.1 and following that theorem, that

$$K_{\ell,\Delta}(f, t^{2\ell})_B \approx \|(B_t - I)^\ell f\|_{X^*}$$

and

$$\|K^*_{\ell,\Delta}(f, t^{2\ell})\|_B \approx \|(B_t - I)^\ell f\|_B.$$

We prefer combinations, as done here, to iterations since in this case we do not average more than once. The corollaries for S_t of iterations would be, under the conditions of Theorem 5.7

$$K^*_{\ell,\Delta}(f, t^{2\ell})_B \approx \sup_{0 \leqslant t_i \leqslant t} \|(S_{t_1} - I) \cdots (S_{t_\ell} - I) f\|_B$$

and

$$K_{\ell,\Delta}(f, t^{2\ell})_B \approx \sup_{0 \leqslant t_i \leqslant t} \|(S_{t_1} - I) \cdots (S_{t_\ell} - I) f\|_{X^*},$$

that is, we have to take suprema on all t_i independently. We prefer supremum on one t and combinations rather than iterations.

In [Di-Iv, Sect. 9], averages on boxes were treated.

In [Di, II], averages on $2d$ points were compared to $K_\Delta(f, t^2)$. While this is more economical information, it cannot lead to S.C.I. of type A (or B) and the analogous results for combinations were not established.

Another comparison is with the average moduli of smoothness. It was shown at least for $B = L_p$ (see [Pe-Po] for instance) that, for $d=1$,

$$\frac{1}{t}\int_0^t \|\Delta_u^r f\|_B \, du \approx \omega^r(f, t)_B,$$

where Δ_u^r represents symmetric differences. From the result here, it follows that for $d=1$ and $r \leqslant 3$ we have

$$\left\| \frac{1}{t}\int_0^t \Delta_u^{2r} f \, du \right\|_B \approx \omega^{2r}(f, t)_B$$

(and similar results for odd r, $r<1$ and $d=1$ can also be shown). Of course, since the direct result is easy in both cases, the present new equivalence has more in it. Moreover, we have S.C.I. and results for $d \neq 1$ as well.

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